

ANALYTIC VERSION OF CRITICAL Q SPACES AND THEIR PROPERTIES

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ABSTRACT. In this paper, we establish an analytic version of critical spaces $Q_\alpha^\beta(\mathbb{R}^n)$ on unit disc \mathbb{D} , denoted by $Q_p^\beta(\mathbb{D})$. Further we prove a relation between $Q_p^\beta(\mathbb{D})$ and Morrey spaces. By the boundedness of two integral operators, we give the multiplier spaces of $Q_p^\beta(\mathbb{D})$.

1. INTRODUCTION

As a new space between $W^{1,n}(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$, Q -type space has been studied extensively since 1990s. In 1995, on the unit disc \mathbb{D} of the complex plane, R. Aulaskari, J. Xiao and R. Zhao first introduced a class of Möbius invariant analytic function space $Q_p(\mathbb{D})$, $p \in (0, 1)$. The class $Q_p(\mathbb{D})$, $p \in (0, 1)$, can be seen as subspaces and subsets of $BMOA$ and UBC on \mathbb{D} . Since then, many studies on $Q_p(\mathbb{D})$ and their characterizations have been done. We refer the reader to [1], [2], [15], [24] and the references therein.

2000 *Mathematics Subject Classification.* 45P05, 30H.

Key words and phrases. Critical Q spaces, analytic Q space, Morrey space.

This work was supported by NNSF(Grant No. 11171203 and No. 11201280); New Teacher's Fund for Doctor Stations, Ministry of Education No.20114402120003; Guangdong Natural Science Foundation (Grant No. 10151503101000025 and No. S2011040004131); Foundation for Distinguished Young Talents in Higher Education of Guangdong, China, LYM11063.

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In order to generalize $Q_p(\mathbb{D})$ to \mathbb{R}^n , in [8], M. Essen, S. Janson, L. Peng and J. Xiao introduced a class of Q -type spaces of several real variables $Q_\alpha(\mathbb{R}^n)$, $\alpha \in (0, 1)$, which is defined as the set of all measurable functions with

$$\sup_I (l(I))^{2\alpha-n} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x - y|^{n+2\alpha}} dx dy < \infty,$$

where the supremum is taken over all cubes I with edge length $l(I)$ and the edges parallel to the coordinate axes in \mathbb{R}^n . Later, in [7], G. Dafni and J. Xiao established the Carleson measure characterization of $Q_\alpha(\mathbb{R}^n)$. For more information of $Q_\alpha(\mathbb{R}^n)$ and their applications, we refer the reader to [7], [8] and [24]. For the generalization of $Q_\alpha(\mathbb{R}^n)$, we refer to [13] and [26].

It is easy to see that Q -type spaces has a structure similar to $BMO(\mathbb{R}^n)$. Moreover, the definition of $Q_\alpha(\mathbb{R}^n)$ implies that the functions in $Q_\alpha(\mathbb{R}^n)$ should own some regularity. Hence, in many cases, $Q_\alpha(\mathbb{R}^n)$ can be seen as an adequate replacement of $BMO(\mathbb{R}^n)$. In recent years, these spaces have been applied to the study of partial differential equations by several authors. In [21], J. Xiao got the well-posedness of the Navier-Stokes equation with data in $Q_\alpha^{-1}(\mathbb{R}^n)$, where $Q_\alpha^{-1}(\mathbb{R}^n) = \nabla \cdot (Q_\alpha(\mathbb{R}^n))^n$. For $\alpha \in (-\frac{n}{2}, 0)$, $Q_\alpha(\mathbb{R}^n) = BMO(\mathbb{R}^n)$. Hence Xiao's result generalizes that of Koch and Tataru [12]. Inspired the results of [12] and [21], P. Li and Z. Zhai introduced a new critical Q -type spaces $Q_\alpha^\beta(\mathbb{R}^n)$ with $\alpha > 0$ and $\max\{\frac{1}{2}, \alpha\} < \beta < 1$ such that $\alpha + \beta - 1 \geq 0$. With data in $Q_\alpha^{\beta-1}(\mathbb{R}^n) = \nabla \cdot (Q_\alpha^\beta(\mathbb{R}^n))^n$ being small, in [13], P. Li and Z. Zhai obtained the well-posedness of the generalized Navier-Stokes equations. This result generalizes the well-posed results of [12] and [21] to the fractional cases. Also, the spaces $Q_\alpha^\beta(\mathbb{R}^n)$ can be used to study the well-posedness of quasi-geostrophic equation, see [14].

Now we state the motivation of this paper. In [13], the authors introduced the following Q -type spaces to study the generalized Navier-Stokes equations.

Definition 1. For $0 < \alpha < 1$, $\max\{\alpha, \frac{1}{2}\} < \beta < 1$ and $\alpha + \beta \geq 1$, $\mathcal{Q}_\alpha^\beta(\mathbb{R}^n)$ is defined as the space of all measurable functions with

$$\sup_I (l(I))^{2\alpha-n+2\beta-2} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x - y|^{n+2\alpha-2\beta+2}} dx dy < \infty,$$

where the supremum is taken over all cubes I with edge length $l(I)$ and the edges parallel to the coordinate axes in \mathbb{R}^n .

By a simple computation, we can see that $\mathcal{Q}_\alpha^\beta(\mathbb{R}^n)$ is the spaces which is invariant under the dilation: $f_\lambda(x) = \lambda^{2\beta-1} f(\lambda x)$. In the research of partial differential equations, such spaces is called critical spaces and play an important role in the well-posedness of fluid equations. If $\beta = 1$, $\mathcal{Q}_\alpha^\beta(\mathbb{R}^n)$ becomes $Q_\alpha(\mathbb{R}^n)$ introduced in [8]. It is well-known that when $n = 1$, $Q_\alpha(\mathbb{R}^1)$ can be seen as the boundary value of $Q_p(\mathbb{D})$. It is natural to ask

Question. *What about the analytic version of $\mathcal{Q}_\alpha^\beta(\mathbb{R}^n)$?*

The aim of this paper is to establish an analytic version on the unit disc of $\mathcal{Q}_\alpha^\beta(\mathbb{R}^n)$. Precisely, we prove a Carleson-measure characterization of the spaces $\mathcal{Q}_p^\beta(\mathbb{T})$, the Q type spaces on the unit circle associate to $\mathcal{Q}_\alpha^\beta(\mathbb{R}^n)$. Using this characterization implies that $\mathcal{Q}_p^\beta(\mathbb{D})$, the space of analytic functions in \mathbb{D} which are the Poisson integral of $\mathcal{Q}_p^\beta(\mathbb{T})$ functions, can be seen as a harmonic extension of $\mathcal{Q}_p^\beta(\mathbb{T})$. Proofs of these results are motivated by the idea of [24] by J. Xiao. Then we study properties of the spaces $\mathcal{Q}_p^\beta(\mathbb{D})$. By the ν -derivative of f on \mathbb{D} , we establish a relation between $\mathcal{Q}_p^\beta(\mathbb{D})$ and Morrey spaces $\mathcal{L}^{2,\lambda}$. For the special case $\beta = 1$, such relation has been obtained by

Z. Wu and C. Xie in [18]. Hence our result can be seen as a generalization of that in [18]. Finally, by the boundedness of two integral operators, we obtain the multiplier spaces of $\mathcal{Q}_p^\beta(\mathbb{D})$.

This paper is organized as follows. In Section 2, we state some notations and terminology which will be used in the sequel. In Section 3, we investigate the harmonic extension of $\mathcal{Q}_\alpha^\beta(\mathbb{T})$ and the analytic version of $\mathcal{Q}_\alpha^\beta(\mathbb{R}^n)$. The relation between $\mathcal{Q}_p^\beta(\mathbb{D})$ and Morrey spaces are given in Section 4. Section 5 is devoted to the study of multiplier spaces of $\mathcal{Q}_p^\beta(\mathbb{D})$.

Throughout this paper, for two functions F and G , we say $F \lesssim G$ if there is a positive constant C independent of F and G such that $F \leq CG$. Furthermore, we say $F \approx G$ (that is, F is comparable with G) whenever $F \lesssim G \lesssim F$.

2. NOTATION AND PRELIMINARIES

Let \mathbb{D} and \mathbb{T} denote respectively the open unit disc and the unit circle in the complex plane \mathbb{C} . Let $H(\mathbb{D})$ be the space of all analytic functions on \mathbb{D} . For $0 < p < \infty$, $\mathcal{Q}_p(\mathbb{D})$ is the set of all functions $f \in H(\mathbb{D})$ with

$$\|f\|_{\mathcal{Q}_p(\mathbb{D})} = |f(0)| + \sup_{a \in \mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_a(z)|^2)^p dA(z) \right)^{1/2} < \infty,$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius map and $dA(z) = \frac{1}{\pi} dx dy$ is the normalized Lebesgue area measure. If $p = 1$, then $\mathcal{Q}_p(\mathbb{D}) = BMOA(\mathbb{D})$, the John-Nirenberg space of all analytic functions with bounded mean oscillation. That is $f \in BMOA(\mathbb{D})$ if and only if f belongs to Hardy space $H^2(\mathbb{D})$ and satisfies

$$\|f\|_{BMOA} = \sup_{I \subset T} |I|^{-1} \int_I |f(\zeta) - f_I| \frac{|d\zeta|}{2\pi} < \infty,$$

where the supremum is taken over all open subarcs $I \subset \mathbb{T}$ with

$$|I| = \int_I \frac{|d\zeta|}{2\pi}, \text{ and } f_I = |I|^{-1} \int_I f(\zeta) \frac{|d\zeta|}{2\pi}.$$

For $-\infty < p < \infty$, $Q_p(\mathbb{T})$ is the space of all Lebesgue measurable functions $f : \mathbb{T} \rightarrow \mathbb{C}$ with

$$\|f\|_{Q_p(\mathbb{T})} = \sup_{I \subset \mathbb{T}} \left(|I|^{-p} \int_I \int_I \frac{|f(\zeta) - f(\eta)|^2}{|\zeta - \eta|^{2-p}} |d\eta| |d\zeta| \right)^{1/2} < \infty,$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$. For more information on spaces $Q_p(\mathbb{D})$ and $Q_p(\mathbb{T})$, see [15], [23], [24] and [25], for example.

Let $I \subset \mathbb{T}$ be an interval. We define Carleson box as

$$S(I) = \left\{ z \in \mathbb{D} : 1 - |I| \leq |z| < 1, \frac{z}{|z|} \in I \right\}.$$

For $0 < p < \infty$, a positive Borel measure μ on \mathbb{D} is said to be a p -Carleson measure if

$$\sup_{I \subset \mathbb{T}} \frac{\mu(S(I))}{|I|^p} < \infty.$$

If $p = 1$, p -Carleson measure is the classical Carleson measure, see [6] and [9].

The following result on p -Carleson measure is well-known, see [1].

Lemma 1. *For $0 < p < \infty$, a positive Borel measure μ on \mathbb{D} is a p -Carleson measure if and only if*

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left(\frac{1 - |a|^2}{|1 - \bar{a}z|^2} \right)^p d\mu(z) < \infty.$$

3. Harmonic extension

In this section, we establish an analytic version on the unit disc of $Q_\alpha^\beta(\mathbb{R}^n)$. At first, we introduce the definition of Q -type spaces on unit circle associated with $Q_\alpha^\beta(\mathbb{R}^n)$.

Definition 2. Let $0 < p < 1$, $1/2 < \beta < 1$ and $f \in L^2(\mathbb{T})$. We say $f \in \mathcal{Q}_p^\beta(\mathbb{T})$ if

$$\|f\|_{\mathcal{Q}_p^\beta(\mathbb{T})} = \sup_{I \subset \mathbb{T}} \left(|I|^{2\beta-2-p} \int_I \int_I \frac{|f(\zeta) - f(\eta)|^2}{|\zeta - \eta|^{4-p-2\beta}} |d\zeta| |d\eta| \right)^{1/2} < \infty,$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$.

That $\mathcal{Q}_p^\beta(\mathbb{T})$ can be seen as the subspaces of the following BMO type spaces.

Definition 3. Let $1/2 < \beta < 1$. Define $BMO^\beta(\mathbb{T})$ as the space of $f \in L^2(\mathbb{T})$ with

$$\|f\|_{BMO^\beta(\mathbb{T})}^2 = \sup_{I \subset \mathbb{T}} |I|^{4\beta-5} \int_I |f(e^{is}) - f_I|^2 ds < \infty,$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$.

The following properties can be deduced from the definitions of $BMO^\beta(\mathbb{T})$ and $\mathcal{Q}_p^\beta(\mathbb{T})$ immediately (we refer the reader to Theorem 3.2 of [13]).

Property 1. Given $p \in (0, 1)$ and $\beta \in (1/2, 1)$. Then

(i) $\mathcal{Q}_p^\beta(\mathbb{T})$ is increasing in p for a fixed β , i.e.

$$\mathcal{Q}_{p_1}^\beta(\mathbb{T}) \subseteq \mathcal{Q}_{p_2}^\beta(\mathbb{T}), \text{ if } p_1 \leq p_2;$$

(ii) $\mathcal{Q}_p^\beta(\mathbb{T}) \subseteq BMO^\beta(\mathbb{T})$.

Let $d\mu_z(\zeta) = \frac{1-|z|^2}{|\zeta-z|^2} \frac{|d\zeta|}{2\pi}$ be the Poisson measure on \mathbb{T} . In addition, denote

$$\hat{f}(z) = \int_{\mathbb{T}} f(\zeta) d\mu_z(\zeta), \quad z \in \mathbb{D}.$$

Similar to Lemma 7.1.1 of [24], we can obtain the following lemma.

Lemma 2. *Given $p \in (0, 1)$, $\beta \in (\frac{1}{2}, 1)$ and $p + 2\beta > 2$. Let I and J be two arcs on \mathbb{T} centered at $\zeta_0 = e^{is_0}$ with $|J| \geq 3|I|$. If $f \in L^2(\mathbb{T})$, then*

$$\begin{aligned} & \int_{S(I)} |\nabla \hat{f}(z)|^2 (1 - |z|^2)^{p-2+2\beta} dm(z) \\ & \lesssim \int_J \int_J \frac{|f(e^{it}) - f(e^{is})|^2}{|e^{it} - e^{is}|^{4-p-2\beta}} dt ds + |I|^{2\beta+p} \left(\int_{|t| \geq 2|J|/3} \frac{|f(e^{i(t+s_0)}) - f_J|}{t^2} dt \right)^2, \end{aligned}$$

where $\nabla = (2\partial/\partial z, 2\partial/\partial \bar{z})$ stands for the gradient vector.

Proof. Without loss of generality, assume that $\zeta_0 = 1$ and ϕ is a function with $0 \leq \phi \leq 1$ such that

$$\begin{cases} \phi = 1 \text{ on } \frac{J}{3}; \\ \text{supp } \phi \subset \frac{2J}{3}; \\ |\phi(z) - \phi(w)| \lesssim \frac{|z-w|}{|J|} \text{ for all } z, w \in T. \end{cases}$$

Writing $\psi = 1 - \phi$, we then have

$$f = f_J + (f - f_J)\phi + (f - f_J)\psi =: f_1 + f_2 + f_3.$$

Note that $|\nabla \hat{f}_J| = 0$, since f_1 is constant. So $|\nabla \hat{f}|^2$ is dominated by $|\nabla \hat{f}_3|^2$ and $|\nabla \hat{f}_2|^2$. For $z = re^{i\theta}$ in the Carleson box $S(I)$,

$$\begin{aligned} |\nabla \hat{f}_3(re^{i\theta})| & \lesssim \int_0^{2\pi} \frac{|f_3(e^{it})|}{(1-r)^2 + (\theta-t)^2} dt \\ & \lesssim \int_{|t| > 2|J|/3} |f(e^{it}) - f_J| \frac{dt}{t^2}. \end{aligned}$$

So we get

$$\begin{aligned}
& \int_{S(I)} |\nabla \hat{f}_3(z)|^2 (1 - |z|^2)^{p-2+2\beta} dA(z) \\
& \lesssim \int_{S(I)} \left(\int_{|t| > 2|J|/3} |f(e^{it}) - f_J| \frac{dt}{t^2} \right)^2 (1 - |z|^2)^{p-2+2\beta} dA(z) \\
& \lesssim \left(\int_{|t| > 2|J|/3} |f(e^{it}) - f_J| \frac{dt}{t^2} \right)^2 \int_0^{|I|/2} d\theta \int_0^{|I|} r^{p-2+2\beta} r dr \\
& \lesssim |I|^{p+2\beta} \left(\int_{|t| > 2|J|/3} |f(e^{it}) - f_J| \frac{dt}{t^2} \right)^2.
\end{aligned}$$

Now for the integral over $S(I)$ of $|\nabla \hat{f}_2|^2$, by Lemma 6.1.1 of [24],

$$\begin{aligned}
& \int_{S(I)} |\nabla \hat{f}_2(z)|^2 (1 - |z|^2)^{p-2+2\beta} dA(z) \\
& \lesssim \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|f_2(\zeta) - f_2(\eta)|^2}{|\zeta - \eta|^{4-p-2\beta}} |d\zeta| |d\eta| \\
& \lesssim \left(\int_{\zeta \in J, \eta \in J} + \int_{\zeta \notin J, \eta \in \frac{3J}{4}} + \int_{\eta \in J, \zeta \in \frac{3J}{4}} \right) \frac{|f_2(\zeta) - f_2(\eta)|^2}{|\zeta - \eta|^{4-p-2\beta}} |d\zeta| |d\eta| \\
& =: M_1 + M_2 + M_3.
\end{aligned}$$

Since

$$\begin{aligned}
|f_2(\zeta) - f_2(\eta)| &= |(f(\zeta) - f_J)\phi(\zeta) - (f(\eta) - f_J)\phi(\eta)| \\
&= |(f(\zeta) - f(\eta))\phi(\zeta) + f(\eta)\phi(\zeta) - f_J\phi(\eta) - (f(\eta) - f_J)\phi(\eta)| \\
&\lesssim |f(\zeta) - f(\eta)| |\phi(\zeta)| + |f(\eta) - f_J| |\phi(\zeta) - \phi(\eta)| \\
&\lesssim |f(\zeta) - f(\eta)| + |J|^{-1} |\zeta - \eta| |f(\eta) - f_J|,
\end{aligned}$$

we have

$$\begin{aligned}
M_1 &= \int_J \int_J \frac{|f_2(\eta) - f_2(\zeta)|^2}{|\zeta - \eta|^{4-p-2\beta}} |d\zeta| |d\eta| \\
&\lesssim \int_J \int_J \frac{|f(\eta) - f(\zeta)|^2}{|\zeta - \eta|^{4-p-2\beta}} |d\zeta| |d\eta| + \int_J \int_J \frac{1}{|J|^2} \frac{|\zeta - \eta|^2 |f(\eta) - f_J|^2}{|\zeta - \eta|^{4-p-2\beta}} \\
&\lesssim \int_J \int_J \frac{|f(\eta) - f(\zeta)|^2}{|\zeta - \eta|^{4-p-2\beta}} |d\zeta| |d\eta| + \frac{1}{|J|^2} \int_J \int_J |f(\eta) - f_J|^2 |d\eta| |\zeta - \eta|^{p-2+2\beta} |d\zeta| \\
&\lesssim \int_J \int_J \frac{|f(\eta) - f(\zeta)|^2}{|\zeta - \eta|^{4-p-2\beta}} |d\zeta| |d\eta| + \frac{1}{|J|^2} \int_J |f(\eta) - f_J|^2 |J|^{p-1+2\beta} |d\eta|.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{1}{|J|^2} \int_J |f(\eta) - f_J|^2 |J|^{p-1+2\beta} |d\eta| &= \frac{1}{|J|^{3-p-2\beta}} \int_J |f(\eta) - f_J|^2 |d\eta| \\
&\lesssim \frac{1}{|J|^{4-p-2\beta}} \int_J \int_J |f(\zeta) - f(\eta)|^2 |d\zeta| |d\eta| \\
&\lesssim \int_J \int_J \frac{|f(\zeta) - f(\eta)|^2}{|\zeta - \eta|^{4-p-2\beta}} |d\zeta| |d\eta|,
\end{aligned}$$

where we used a fact that

$$f(\eta) - f_J = \frac{1}{|J|} \int_J (f(\eta) - f(\zeta)) d\zeta$$

and $|\zeta - \eta| \leq 2|J|$. We obtain

$$M_1 \lesssim \int_J \int_J \frac{|f(\zeta) - f(\eta)|^2}{|\zeta - \eta|^{4-p-2\beta}} |d\zeta| |d\eta|.$$

Now, we estimate M_2 . Recall that

$$M_2 = \int_{\zeta \notin J} \int_{\eta \in \frac{3J}{4}} \frac{|f_2(\zeta) - f_2(\eta)|}{|\zeta - \eta|^{2-p+2\beta}} |d\zeta| |d\eta|.$$

Since $\zeta \notin J$, then $\phi(\zeta) = 0$ and $f_2(\zeta) = (f(\zeta) - f_J)\phi(\zeta) = 0$. This implies that

$$\begin{aligned}
M_2 &= \int_{\zeta \notin J} \int_{\eta \in \frac{3J}{4}} \frac{|f_2(\eta)|}{|\zeta - \eta|^{2-p+2\beta}} |d\zeta| |d\eta| \\
&= \int_{\eta \in \frac{3J}{4}} |f_2(\eta)|^2 |d\eta| \int_{\zeta \notin J} \frac{|d\zeta|}{|\zeta - \eta|^{4-p-2\beta}}.
\end{aligned}$$

Assume $J = \{e^{i\theta} : |\theta - s_0| \leq h\}$. Since $e^{is_0} = 1$, then $s_0 = 0$. So $|J| = \frac{h}{2\pi}$ and $|\theta| \leq 2\pi J$. Take $\zeta = e^{i\theta} \notin J$. Then $|\theta| > 2\pi J$. If $\eta \in \frac{3J}{4}$, then $\eta = e^{i \arg \eta}$ and $|\arg \eta| \leq \frac{3\pi}{2}|J|$. This leads that $|\zeta - \eta| \geq \frac{\pi}{2}|J|$. Hence

$$\begin{aligned} M_2 &\lesssim \int_{\eta \in \frac{3J}{4}} |f_2(\eta) - f_J|^2 \frac{|d\eta|}{|J|^{3-p-2\beta}} \\ &\lesssim \frac{1}{|J|^{4-p-2\beta}} \int_{\zeta \notin J} \int_{\eta \in \frac{3J}{4}} |f(\zeta) - f(\eta)|^2 |d\zeta| |d\eta|. \end{aligned}$$

For M_3 , the proof is similarly and is so omitted. \square

Theorem 1. *Given $0 < p < 1$, $\beta \in (1/2, 1)$ and $p - 2 + 2\beta > 0$. Let $f \in BMO^\beta(\mathbb{T})$. The following conditions are equivalent:*

(1) $f \in Q_p^\beta(\mathbb{T})$;

(2)

$$\sup_{I \subset \mathbb{T}} |I|^{2\beta-p-2} \int_0^{|I|} \left(\int_I |f(e^{i(s+t)}) - f(e^{is})|^2 ds \right) \frac{dt}{t^{4-p-2\beta}} < \infty,$$

where the supremum is taken over all arcs $I \subseteq \mathbb{T}$;

(3) $|\nabla \hat{f}(z)|^2 (1 - |z|^2)^{p-2+2\beta} dA(z)$ is a $(p + 2 - 2\beta)$ -Carleson measure.

Proof. We give the proof according the following order (3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (3).

(3) \Rightarrow (2). Without loss of generality, assume that I is the interval $(0, |I|)$

with $|I| < \frac{1}{4}$. If $t \in (0, |I|)$, then

$$\begin{aligned} \left(\int_{3I} |f(e^{i(v+t)}) - f(e^{iv})|^2 dv \right)^{1/2} &\leq 2 \int_{1-t}^1 \left(\int_{4I} \left| \frac{\partial \hat{f}}{\partial n}(ue^{is}) \right|^2 ds \right)^{1/2} du \\ &\quad + 2 \int_0^t \left(\int_{4I} \left| \frac{\partial \hat{f}}{\partial \theta}((1-t)e^{is}) \right|^2 ds \right)^{1/2} du \\ &=: I_1 + I_2. \end{aligned}$$

Since

$$\begin{aligned} \int_0^{|I|} \frac{I_1^2}{t^{4-p-2\beta}} dt &\lesssim \int_0^{|I|} t^{p+2\beta-2} \left(\int_{4I} \left| \frac{\partial \hat{f}}{\partial n}((1-t)e^{is}) \right|^2 ds \right) dt \\ &\lesssim \int_{S(4I)} |\nabla \hat{f}(z)|^2 (1-|z|)^{p+2\beta-2} dA(z), \end{aligned}$$

and

$$\begin{aligned} \int_0^{|I|} \frac{I_2^2}{t^{4-p-2\beta}} dt &\lesssim \int_0^{|I|} t^{p+2\beta-2} \left(\int_{4I} \left| \frac{\partial \hat{f}}{\partial \theta}(re^{is}) \right|^2 ds \right) dt \\ &\lesssim \int_{S(4I)} |\nabla \hat{f}(z)|^2 (1-|z|)^{p+2\beta-2} dA(z). \end{aligned}$$

We have

$$\begin{aligned} &\frac{1}{|I|^{p+2-2\beta}} \int_0^{|I|} \left(\int_I |f(e^{i(\theta+t)}) - f(e^{i\theta})|^2 d\theta \right) \frac{dt}{t^{4-p-2\beta}} \\ &\lesssim \frac{1}{|I|^{p+2-2\beta}} \int_0^{|I|} \int_{3I} |f(e^{i(\theta+t)}) - f(e^{i\theta})|^2 d\theta \frac{dt}{t^{4-p-2\beta}} \\ &\lesssim \frac{1}{|I|^{p+2-2\beta}} \int_0^{|I|} (I_1^2 + I_2^2) \frac{dt}{t^{4-p-2\beta}} \\ &\lesssim \frac{1}{|I|^{p+2-2\beta}} \int_{S(4I)} |\nabla \hat{f}(z)|^2 (1-|z|)^{p+2\beta-2} dA(z) \leq C, \end{aligned}$$

where we used a fact that $|\nabla \hat{f}(z)|^2 (1-|z|)^{p+2\beta-2} dA(z)$ is a $(p+2-2\beta)$ -Carleson measure.

(2) \Rightarrow (1). Without loss of generality, we assume that I is a small subarc of \mathbb{T} (say $|I| \leq 1/4$) and assume that $I = (a, b) \subset [0, 2\pi]$. We get

$$\begin{aligned} \int_I \int_I \frac{|f(e^{is}) - f(e^{it})|^2}{|e^{is} - e^{it}|^{4-p-2\beta}} ds dt &\lesssim \int_a^b \int_{a < s+t < b} \frac{|f(e^{i(s+t)}) - f(e^{is})|^2}{|t|^{4-p-2\beta}} dt ds \\ &= \int_a^b \left(\int_{a-s}^0 + \int_0^{b-s} \right) \frac{|f(e^{i(s+t)}) - f(e^{is})|^2}{|t|^{4-p-2\beta}} dt ds \\ &\lesssim \int_0^{b-a} \frac{1}{\theta^{4-p-2\beta}} \left(\int_I |f(e^{i(s+\theta)}) - f(e^{is})|^2 ds \right) d\theta \\ &\lesssim |I|^{p+2-2\beta}. \end{aligned}$$

Hence $f \in Q_p^\beta(\mathbb{T})$.

(1) \Rightarrow (3). Let $e^{is}, e^{it} \in I$, then $|e^{is} - e^{it}| \leq |I|$. We have

$$\begin{aligned} \int_I \int_I |f(e^{is}) - f(e^{it})|^2 ds dt &\leq |I|^{4-p-2\beta} \int_I \int_I \frac{|f(e^{is}) - f(e^{it})|^2}{|e^{is} - e^{it}|^{4-p-2\beta}} ds dt \\ &\lesssim |I|^{6-4\beta} \|f\|_{Q_p^\beta(\mathbb{T})}^2. \end{aligned}$$

So

$$\begin{aligned} \int_I |f(e^{is}) - f_I| ds &\leq \int_I \frac{1}{|I|} \int_I |f(e^{is}) - f(e^{it})| dt ds \\ &\leq \frac{1}{|I|} \int_I \int_I |f(e^{is}) - f(e^{it})|^2 dt ds \\ &\lesssim |I|^{5-4\beta} \|f\|_{Q_p^\beta(\mathbb{T})}^2. \end{aligned}$$

Let $J = 3I$ and $|I| < 1/3$. Using Lemma 2 (for $\zeta_0 = 0$), we only need to show

$$\int_{|t| \geq 2|J|/3} |f(e^{i(t+s_0)}) - f_J| \frac{dt}{t^2} \lesssim |I|^{1-2\beta} \|f\|_{BMO^\beta(T)}. \quad (3.1)$$

It is easy to see that

$$\begin{aligned} \int_{|t| \geq |J|/3} |f(e^{it}) - f_J| \frac{dt}{t^2} &\leq \sum_{k=1}^{\infty} \int_{3^{k-1}|J| < |t| \leq 3^k|J|} |f(e^{it}) - f_J| \frac{dt}{t^2} \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{(3^k|J|)^2} \int_{|t| \leq 3^k|J|} |f(e^{it}) - f_J| dt \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{(3^k|J|)^2} \int_{|t| \leq 3^k|J|} |f(e^{it}) - f_{3^{k+1}J}| dt + \sum_{k=1}^{\infty} \frac{|f_{3^{k+1}J} - f_J|}{3^k|J|} \\ &=: M_1 + M_2. \end{aligned} \quad (3.2)$$

At first, we estimate the term M_1 . Since

$$\begin{aligned}
M_1 &= \sum_{k=1}^{\infty} \frac{1}{(3^k|J|)^2} \int_{|t| \leq 3^k|J|} |f(e^{it}) - f_{3^{k+1}J}| dt \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{(3^k|J|)} \left(\frac{1}{(3^k|J|)} \int_{|t| \leq 3^k|J|} |f(e^{it}) - f_{3^{k+1}J}|^2 dt \right)^{1/2} \\
&= \sum_{k=1}^{\infty} (3^k|J|)^{1-2\beta} \|f\|_{BMO^\beta(\mathbb{T})} \\
&\lesssim |J|^{1-2\beta} \|f\|_{BMO^\beta(\mathbb{T})}.
\end{aligned} \tag{3.3}$$

We next deal with the term M_2 . By the definition of $BMO^\beta(\mathbb{R}^n)$, for any $I, J \subset \mathbb{T}$ with $I \subset J$ and $|J| \leq 3|I|$,

$$\begin{aligned}
|f_I - f_J| &\lesssim \frac{1}{|I|} \int_I |f(e^{is}) - f_J| ds \\
&\lesssim \left(\frac{1}{|I|} \int_I |f(e^{is}) - f_J|^2 ds \right)^{1/2} \\
&\lesssim |J|^{2-2\beta} \|f\|_{BMO^\beta(\mathbb{T})}.
\end{aligned}$$

Since $\beta > 1/2$, we obtain

$$\begin{aligned}
M_2 &= \sum_{k=1}^{\infty} \frac{|f_{3^{k+1}J} - f_J|}{3^k|J|} \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{3^k|J|} (|f_{3^{k+1}J} - f_{3^k J}| + |f_{3^k J} - f_{3^{k-1}J}| \cdots + |f_{3J} - f_J|) \\
&\lesssim \sum_{k=1}^{\infty} \frac{k+1}{3^k|J|} (3^{k+1}|J|)^{2-2\beta} \|f\|_{BMO^\beta(\mathbb{T})} \\
&\lesssim |I|^{1-2\beta} \|f\|_{BMO^\beta(\mathbb{T})}.
\end{aligned} \tag{3.4}$$

So (3.1) follows from (3.2), (3.3) and (3.4). The proof of this theorem is completed by Lemma 2. \square

Let $C^1(\mathbb{D})$ be the space of complex value functions which is continuously differentiable on \mathbb{D} . Then we have the following corollary.

Corollary 1. *Given $0 < p < 1$, $1/2 < \beta < 1$ and $p + 2\beta > 2$. Let F be a function defined on $\overline{\mathbb{D}}$ such that $F \in C^1(\mathbb{D})$ and $F|_{\mathbb{T}} = f$. If $|\nabla F(z)|^2(1 - |z|^2)^p dm(z)$ is a $(p + 2 - 2\beta)$ -Carleson measure, then $f \in \mathcal{Q}_p^\beta(\mathbb{T})$.*

It is easy to see that $|\nabla \hat{f}|^2 = 4|f'(z)|$ when $f \in H(\mathbb{D})$. Theorem 1 suggests us to define $\mathcal{Q}_p^\beta(\mathbb{D})$ spaces as following.

Definition 4. For $0 < p < 1$ and $1/2 < \beta \leq 1$. The $\mathcal{Q}_p^\beta(\mathbb{D})$ stands for the space of all $f \in H(\mathbb{D})$ satisfying

$$\sup_{|I| \subset \mathbb{T}} \frac{1}{|I|^{p+2-2\beta}} \int_{S(I)} |f'(z)|^2 (1 - |z|^2)^{p-2+2\beta} dA(z) < \infty.$$

Theorem 1 gives the answer to the Question. In fact, the $\mathcal{Q}_p^\beta(\mathbb{D})$ is the space of those analytic functions in \mathbb{D} which are the Poisson integral of a function ψ with $\psi \in \mathcal{Q}_p^\beta(\mathbb{T})$ (for related result, see Theorem 3.1 of [11] and Proposition 5.1 of [10]).

Remark 1. In fact, by Lemma 1, we can prove that $\mathcal{Q}_p^\beta(\mathbb{D})$ can be defined equivalently as

$$\|f\|_{\mathcal{Q}_p^\beta(\mathbb{D})} = |f(0)| + \sup_{a \in \mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{4\beta-4} (1 - |\sigma_a(z)|^2)^{p+2-2\beta} dA(z) \right)^{1/2}.$$

4. $\mathcal{Q}_p^\beta(\mathbb{D})$ and Morrey spaces

In this section, we establish a relation between $\mathcal{Q}_p^\beta(\mathbb{D})$ and Morrey spaces. The Morrey spaces are defined as follows.

Definition 5. Given $0 < \lambda \leq 1$. The Morrey space $\mathcal{L}^{2,\lambda}(\mathbb{D})$ is defined as the set of all f which belong to Hardy space $H^2(\mathbb{D})$ and satisfy

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|^\lambda} \int_I |f(\zeta) - f_I|^2 \frac{|d\zeta|}{2\pi} < \infty,$$

where the supremum is taken over all open subarcs $I \subset \mathbb{T}$.

Clearly, $\mathcal{L}^{2,1}(\mathbb{D}) = BMOA(\mathbb{D})$.

In [19], Z. Wu proved a Carleson measure characterization of Morrey spaces. A similar result on the half plane can be found in [18, Theorem 1].

Lemma 3. (Lemma 4.1 of [19]) *Let $0 < \lambda \leq 1$ and $f \in H(\mathbb{D})$. Then f belongs to $\mathcal{L}^{2,\lambda}(\mathbb{D})$ if and only if $|f'(z)|^2(1 - |z|^2)dA(z)$ is a λ -Carleson measure.*

For the proof of the main result of this section, we need the following technical lemma which can be compared with Theorem 3 of [18]. For a fixed $b > 1$ and a measurable function ψ , define the linear operator T_σ as

$$T_\sigma \psi(w) = \int_{\mathbb{D}} \frac{(1 - |z|^2)^{b-1}}{|1 - \bar{z}w|^{b+\sigma}} \psi(z) dA(z), \quad \sigma > 0.$$

Lemma 4. *Let $0 < p \leq 1$, $0 < \beta \leq 1$, $\sigma > \max\{(3 - p - 2\beta)/2, 2 - 2\beta\}$, $p + 2\beta > 2$ and ψ be a measurable function on \mathbb{D} . If $|\psi(z)|^2(1 - |z|^2)^{p-2+2\beta}dA(z)$ is a $(p + 2 - 2\beta)$ -Carleson measure, then $|T_\sigma \psi(z)|^2(1 - |z|^2)^{2\sigma+p+2\beta-4}dA(z)$ is also a $(p + 2 - 2\beta)$ -Carleson measure.*

Proof. Given a subarc $I \subset T$ and any positive integer $n \leq \log_2(1/|I|)$. We have

$$\begin{aligned}
& \int_{S(I)} |T_\sigma \psi(w)|^2 (1 - |w|^2)^{2\sigma+p+2\beta-4} dA(w) \\
& \leq \int_{S(I)} \left(\int_{\mathbb{D}} \frac{(1 - |z|^2)^{b-1}}{|1 - \bar{z}w|^{b+\sigma}} |\psi(z)| dA(z) \right)^2 (1 - |w|^2)^{2\sigma+p+2\beta-4} dA(w) \\
& = \int_{S(I)} \left(\left(\int_{S(2I)} + \int_{\mathbb{D} \setminus S(2I)} \right) \frac{(1 - |z|^2)^{b-1}}{|1 - \bar{z}w|^{b+\sigma}} |\psi(z)| dA(z) \right)^2 (1 - |w|^2)^{2\sigma+p+2\beta-4} dA(w) \\
& \leq 2 \int_{S(I)} \left(\int_{S(2I)} \frac{(1 - |z|^2)^{b-1} (1 - |w|^2)^{\sigma+\frac{p}{2}+\beta-2}}{|1 - \bar{z}w|^{b+\sigma}} |\psi(z)| dA(z) \right)^2 dA(w) \\
& \quad + 2 \int_{S(I)} \left(\int_{\mathbb{D} \setminus S(2I)} \frac{(1 - |z|^2)^{b-1} (1 - |w|^2)^{\sigma+\frac{p}{2}+\beta-2}}{|1 - \bar{z}w|^{b+\sigma}} |\psi(z)| dA(z) \right)^2 dA(w) \\
& =: M_1 + M_2.
\end{aligned}$$

Let

$$K(w, z) = \frac{(1 - |z|^2)^{b-\frac{p}{2}-\beta} (1 - |w|^2)^{\sigma+\frac{p}{2}+\beta-2}}{|1 - \bar{z}w|^{b+\sigma}}.$$

Consider the operator

$$Bf(w) = \int_{\mathbb{D}} f(z) K(w, z) dA(z), \quad f \in L^2(\mathbb{D}).$$

For $z \in \mathbb{D}$, using Lemma 3.10 of [30], we have

$$\int_{\mathbb{D}} K(w, z) (1 - |w|^2)^{-1/2} dA(w) \lesssim (1 - |z|^2)^{-1/2}$$

and

$$\int_{\mathbb{D}} K(w, z) (1 - |z|^2)^{-1/2} dA(z) \lesssim (1 - |w|^2)^{-1/2}.$$

By Schur's lemma ([30, Corollary 3.7]), B is bounded on $L^2(\mathbb{D})$.

Take

$$g(z) = (1 - |z|^2)^{\frac{p}{2}+\beta-1} |\psi(z)| \chi_{S(2I)}(z).$$

Since $|\psi(z)|^2(1 - |z|^2)^{p+2\beta-2}dA(z)$ is a $(p + 2 - 2\beta)$ -Carleson measure, then $g \in L^2(\mathbb{D})$ and $\|g\|_{L^2}^2 \lesssim |I|^{p+2-2\beta}$. Thus,

$$\begin{aligned} M_1 &\lesssim \int_{\mathbb{D}} \left| \int_{\mathbb{D}} K(w, z)g(z)dA(z) \right|^2 dA(w) \\ &= \|B(g)\|_{L^2}^2 \lesssim \|g\|_{L^2}^2 \lesssim |I|^{p+2-2\beta}. \end{aligned}$$

Write

$$D \setminus S(2I) = \bigcup_{n=1}^{\infty} S(2^{n+1}I) \setminus S(2^n I) =: \bigcup_{n=1}^{\infty} \Delta_n.$$

For $n \geq 0$, the following inequality is well-known ([9, p.232]):

$$|1 - \bar{z}w| \geq C2^n |I|,$$

where $w \in S(I)$ and $z \in S(2^{n+1}I) \setminus S(2^n I)$. Note that

$$\int_{S(2^n I)} (1 - |w|^2)^a dA(w) \lesssim (2^n |I|)^{a+2}, \quad n \geq 0, a > -1.$$

Using Hölder's inequality, we have

$$\begin{aligned} \int_{S(2^{n+1}I)} |\psi(z)|(1 - |z|^2)^{b-1} dA(z) &\leq \left(\int_{S(2^{n+1}I)} |\psi(z)|^2(1 - |z|^2)^{p+2\beta-2} dA(z) \right)^{1/2} \\ &\quad \times \left(\int_{S(2^{n+1}I)} (1 - |z|^2)^{2b-p-2\beta} dA(z) \right)^{1/2} \\ &\leq \left(\int_{S(2^{n+1}I)} |\psi(z)|^2(1 - |z|^2)^{p+2\beta-2} dA(z) \right)^{1/2} \\ &\quad \times (2^{n+1}|I|)^{b-\beta+1-p/2}. \end{aligned}$$

We get

$$\begin{aligned}
M_2 &= 2 \int_{S(I)} \left(\sum_{n=1}^{\infty} \int_{\Delta_n} \frac{(1 - |z|^2)^{b-1}}{|1 - \bar{z}w|^{b+\sigma}} |\psi(z)| dA(z) \right)^2 (1 - |w|^2)^{2\sigma+p+2\beta-4} dA(w) \\
&\lesssim \int_{S(I)} \left(\sum_{n=1}^{\infty} \frac{1}{(2^n |I|)^{b+\sigma}} \int_{S(2^{n+1}I)} |\psi(z)|(1 - |z|^2)^{b-1} dA(z) \right)^2 (1 - |w|^2)^{2\sigma+p+2\beta-4} dA(w) \\
&\lesssim |I|^{2\sigma+p+2\beta-2} \left(\sum_{n=1}^{\infty} \frac{1}{(2^n |I|)^{b+\sigma}} \int_{S(2^{n+1}I)} |\psi(z)|(1 - |z|^2)^{b-1} dA(z) \right)^2 \\
&\lesssim |I|^{p-2\beta+2} \left(\sum_{n=1}^{\infty} \frac{1}{2^{n(\sigma+2\beta-2)}} \left(\frac{1}{(2^{n+1}|I|)^{p-2\beta+2}} \int_{S(2^{n+1}I)} |\psi(z)|^2 (1 - |z|^2)^{p+2\beta-2} dA(z) \right)^{1/2} \right)^2 \\
&\lesssim |I|^{p-2\beta+2} \left(\sum_{n=1}^{\infty} \frac{1}{2^{n(\sigma+2\beta-2)}} \right)^2 \\
&\lesssim |I|^{p-2\beta+2},
\end{aligned}$$

where we used the fact that $|\psi(z)|^2 (1 - |z|^2)^{p-2+2\beta} dA(z)$ is a $(p + 2 - 2\beta)$ -Carleson measure and $\sigma > 2 - 2\beta$. This completes the proof. \square

Remark 2. If $\beta = 1$, the above lemma is Theorem 3 of [18].

Let Γ be the gamma function and $[\nu]$ the smallest integer bigger than or equal to ν ($\nu > 0$). For fixed $b > 1$, consider the fractional derivative, called ν -derivative of f ,

$$f^{(\nu)}(z) = \frac{\Gamma(b + \nu)}{\Gamma(b)} \int_{\mathbb{D}} \frac{\bar{w}^{[\nu-1]} f'(w)}{(1 - \bar{w}z)^{b+\nu}} (1 - |w|^2)^{b-1} dA(w).$$

If ν is a positive integer, then ν -derivative is just the usual ν -th order derivative. But $f^{(\nu)}$ depends on b if ν is not an integer (see, for example, [18]).

Theorem 2. Suppose that $1/2 < \beta \leq 1$, $2 - 2\beta < p \leq 1$ and $\nu > \max\{(3 - p - 2\beta)/2, 2 - 2\beta\}$. Let $f \in H(\mathbb{D})$. Then f belongs to $Q_p^\beta(\mathbb{D})$ if and only if $|f^{(\nu)}(z)|^2 (1 - |z|^2)^{2\nu+p+2\beta-4} dA(z)$ is a $(p + 2 - 2\beta)$ -Carleson measure.

Proof. At first, if $f \in \mathcal{Q}_p^\beta(\mathbb{D})$, then $|f'(z)|^2(1 - |z|^2)^{p-2+2\beta}$ is a $(p + 2 - 2\beta)$ -Carleson measure. Note that $\nu > \max\{(3 - p - 2\beta)/2, 2 - 2\beta\}$ and

$$|f^{(\nu)}(z)| \leq \frac{\Gamma(b + \nu)}{\Gamma(b)} \int_{\mathbb{D}} \frac{(1 - |w|^2)^{b-1}}{|1 - \overline{w}z|^{b+\nu}} |f'(w)| dA(w) = \frac{\Gamma(b + \nu)}{\Gamma(b)} T_\nu |f'(z)|.$$

The desired result follows from Lemma 4.

Conversely, if $|f^{(\nu)}(z)|^2(1 - |z|^2)^{2\nu+p+2\beta-4} dA(z)$ is a $(p + 2 - 2\beta)$ -Carleson measure, then

$$\int_{\mathbb{D}} |f^{(\nu)}(z)|^2 (1 - |z|^2)^{2\nu+p+2\beta-4} dA(z) < \infty.$$

Combining this with Hölder's inequality yields

$$\begin{aligned} \int_{\mathbb{D}} |f^{(\nu)}(z)| (1 - |z|^2)^\nu dA(z) &\leq \left(\int_{\mathbb{D}} |f^{(\nu)}(z)|^2 (1 - |z|^2)^{2\nu} dA(z) \right)^{1/2} \\ &\leq \left(\int_{\mathbb{D}} |f^{(\nu)}(z)|^2 (1 - |z|^2)^{2\nu+p+2\beta-4} dA(z) \right)^{1/2} \\ &< \infty. \end{aligned}$$

For $m = [\nu - 1]$, $b > 1$ and $z \in \mathbb{D}$, we know

$$\int_{\mathbb{D}} \frac{(1 - |w|^2)^{b-2}}{|1 - \overline{w}z|^{b+m+1}} dA(w) < \infty.$$

Let $\psi(z) = |f^{(\nu)}(z)|(1 - |z|^2)^{\nu-1}$. Then $|\psi(z)|^2(1 - |z|^2)^{p-2+2\beta} dA(z)$ is a $(p + 2 - 2\beta)$ -Carleson measure. We next prove that

$$|f^{m+1}(z)| \lesssim |T_{m+1}\psi(z)|.$$

In fact, for $\lambda > 0$,

$$\frac{1}{(1 - \overline{w}z)^\lambda} = \sum_{j=0}^{\infty} \frac{\Gamma(j + \lambda)}{\Gamma(j + 1)\Gamma(\lambda)} (\overline{w}z)^j.$$

It is easy to check that

$$(z^k)^{(\nu)} = \begin{cases} 0 & \text{if } k < [\nu - 1] + 1; \\ \frac{\Gamma(k+1)\Gamma(b+k+\nu-1-[\nu-1])}{\Gamma(b+k)\Gamma(k-[\nu-1])} z^{k-1-[\nu-1]} & \text{if } k \geq [\nu - 1] + 1. \end{cases}$$

Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$. A direct computation shows that $f^{(\nu)}(z) = \sum_{j=0}^{\infty} a_j^{(\nu)} z^j$, where

$$a_j^{(\nu)} = a_{j+m+1} \frac{\Gamma(j+b+\nu)\Gamma(j+m+2)}{\Gamma(j+1)\Gamma(j+m+b+1)}.$$

Now applying Parseval's theorem ([17]), we obtain

$$\begin{aligned} & \frac{\Gamma(b+m+1)}{\Gamma(b+\nu-1)} \int_{\mathbb{D}} \frac{(1-|w|^2)^{b-2}}{(1-\bar{w}z)^{b+m+1}} f^{(\nu)}(w) (1-|w|^2)^{\nu} dA(w) \\ &= \frac{\Gamma(b+m+1)}{\Gamma(b+\nu-1)} \int_{\mathbb{D}} (1-|w|^2)^{b+\nu-2} \left(\sum_{k=0}^{\infty} \frac{\Gamma(k+b+m+1)}{k!\Gamma(b+m+1)} \bar{w}^k z^k \right) \left(\sum_{j=0}^{\infty} a_j^{(\nu)} w^j \right) dA(w) \\ &= \frac{\Gamma(b+m+1)}{\Gamma(b+\nu-1)} \sum_{k=0}^{\infty} a_k^{(\nu)} z^k \frac{\Gamma(k+b+m+1)}{k!\Gamma(b+m+1)} \int_{\mathbb{D}} (1-|w|^2)^{b+\nu-2} |w|^{2k} dA(w) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(k+m+2)}{\Gamma(k+1)} a_{k+m+1} z^k \\ &= f^{(m+1)}(z). \end{aligned}$$

Hence $|f^{(m+1)}| \lesssim |T_{m+1}\psi(z)|$. Using Lemma 4 again yields that $|f^{(m+1)}|^2(1-|z|^2)^{2m+p+2\beta-2} dA(z)$ is a $(p+2-2\beta)$ -Carleson measure. So the desired result follows from Theorem 4.33 of [16]. \square

Based on Theorem 2, we obtain the main result of this section.

Theorem 3. *Let $0 < p \leq 1$, $1/2 < \beta \leq 1$ with $2\beta - p \geq 1$. Then $f \in \mathcal{Q}_p^{\beta}(\mathbb{D})$ if and only if $f^{(\frac{3-p-2\beta}{2})} \in \mathcal{L}^{2,p-2\beta+2}(\mathbb{D})$.*

Proof. For $\nu > 0$, a simple calculation shows that $(f^{(\nu)})' = f^{(\nu+1)}$. If $f \in Q_p^\beta(\mathbb{D})$, by Theorem 2,

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|^{p+2-2\beta}} \int_{S(I)} |f^{(\frac{5-p-2\beta}{2})}(z)|^2 (1 - |z|^2) dA(z) < \infty.$$

From Lemma 3, we get $f^{(\frac{3-p-2\beta}{2})} \in \mathcal{L}^{2,p-2\beta+2}(\mathbb{D})$.

On the other hand, if $f^{(\frac{3-p-2\beta}{2})} \in \mathcal{L}^{2,p-2\beta+2}$. Using Lemma 3 again, we know that $|f^{(\frac{5-p-2\beta}{2})}(z)|^2 (1 - |z|^2) dA(z)$ is a $(p + 2 - 2\beta)$ -Carleson measure. Note that $\frac{5-p-2\beta}{2} > \max\{\frac{3-p-2\beta}{2}, 2 - 2\beta\}$, Theorem 2 implies $f \in Q_p^\beta(\mathbb{D})$. We finish the proof. \square

Remark 3. If $\beta = 1$, Theorem 3 yields that $f \in Q_p(\mathbb{D})$ if and only if $f^{(\frac{1-p}{2})} \in \mathcal{L}^{2,p}(\mathbb{D})$. This result on the upper half plane was obtained in [18]. The relationship between Q_K and Morrey type spaces is established in [20]. Moreover, Theorem 3 can be compared with Theorem 3.2 of [20].

5. Multiplier spaces of $Q_p^\beta(\mathbb{D})$

In this section, we consider the multiplier spaces of $Q_p^\beta(\mathbb{D})$.

Definition 6. Let X be a Banach space of analytic functions on \mathbb{D} .

- (1) An analytic function $g \in H(\mathbb{D})$ is called a pointwise multiplier on X if $gX \subset X$.
- (2) The multiplier space of X , denoted by $\mathcal{M}(X)$, is the set of all pointwise multipliers on X .

In the study of multiplier spaces, one useful tool is the following multiplication operator induced by g :

$$M_g f = gf, \text{ where } f \in X.$$

In fact, by the Closed Graph Theorem, it can be proved that M_g is a bounded linear operator on X if and only if $g \in \mathcal{M}(X)$. We refer the reader to [5] for more information about pointwise multipliers.

To study M_g , we need the following integral operators. For any $g \in H(\mathbb{D})$,

(1) Volterra type operator T_g induced by g is defined as:

$$T_g f(z) = \int_0^z f(w)g'(w)dw, \quad f \in H(\mathbb{D}).$$

(2) The operator I_g related to T_g is defined as:

$$I_g f(z) = \int_0^z f'(w)g(w)dw, \quad f \in H(\mathbb{D});$$

Remark 4. Generally, g is assumed to be non-constant. In fact, if g is a constant C , then $I_g = C(f(z) - f(0))$ and $T_g = 0$.

It is easy to see that the multiplication operator M_g can be decomposed as

$$M_g f(z) = f(0)g(0) + I_g f(z) + T_g f(z).$$

Hence the boundedness of M_g can be deduced from that of I_g and T_g . We next study the boundedness of the operators I_g and T_g on $\mathcal{Q}_p^\beta(\mathbb{D})$. We need the following lemma ([28, Lemma 1]).

Lemma 5. Suppose that $s > -1$, $r, t > 0$ and $t < s + 2 < r$. We have

$$\int_{\mathbb{D}} \frac{(1 - |z|^2)^s}{|1 - \bar{b}z|^r |1 - \bar{a}z|^t} dA(z) \lesssim \frac{1}{(1 - |b|^2)^{r-s-2} |1 - \bar{a}b|^t},$$

where $a, b \in \mathbb{D}$.

Lemma 6. Let $0 < p \leq 1$ and $1/2 < \beta \leq 1$. For $b \in \mathbb{D}$, the function

$$f_b(z) = (1 - |b|^2)^{2-2\beta} (\sigma_b(z) - b)$$

belongs to $\mathcal{Q}_p^\beta(\mathbb{D})$. Moreover, $\|f_b\|_{\mathcal{Q}_p^\beta(\mathbb{D})} \lesssim 1$.

Proof. Applying Lemma 5 for $t = p + 2 - 2\beta$, $r = 4$ and $s = p + 2\beta - 2$, we get

$$\begin{aligned}
& \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'_b(z)|^2 (1 - |z|^2)^{4\beta-4} (1 - |\sigma_a(z)|^2)^{p+2-2\beta} dA(z) \\
&= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |b|^2)^{6-4\beta} (1 - |z|^2)^{p+2\beta-2} (1 - |a|^2)^{p+2-2\beta}}{|1 - \bar{b}z|^4 |1 - \bar{a}z|^{2(p+2-2\beta)}} dA(z) \\
&\lesssim (1 - |b|^2)^{6-4\beta} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |z|^2)^{p+2\beta-2}}{|1 - \bar{b}z|^4 |1 - \bar{a}z|^{p+2-2\beta}} dA(z) \\
&\lesssim \frac{(1 - |b|^2)^{2-2\beta+p}}{|1 - \bar{a}b|^{p+2-2\beta}} \\
&\lesssim 1.
\end{aligned}$$

So $f_b(z) \in \mathcal{Q}_p^\beta(\mathbb{D})$ and $\|f_b(z)\|_{\mathcal{Q}_p^\beta(\mathbb{D})} \lesssim 1$. □

Theorem 4. For $0 < p \leq 1$ and $1/2 < \beta \leq 1$. Let $g \in H(\mathbb{D})$. Then I_g is bounded on $\mathcal{Q}_p^\beta(\mathbb{D})$ if and only if $g \in H^\infty$. Moreover, $\|I_g\| \approx \sup_{z \in \mathbb{D}} |g(z)|$.

Proof. It is easy to see that

$$\begin{aligned}
\|I_g f\|_{\mathcal{Q}_p^\beta(\mathbb{D})} &= \sup_{a \in \mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 |g(z)|^2 (1 - |z|^2)^{4\beta-4} (1 - |\sigma_a(z)|^2)^{p+2-2\beta} dA(z) \right)^{1/2} \\
&\leq \sup_{z \in \mathbb{D}} |g(z)| \|f\|_{\mathcal{Q}_p^\beta(\mathbb{D})}.
\end{aligned}$$

That is

$$\|I_g\| \leq \sup_{z \in \mathbb{D}} |g(z)|.$$

On the other hand, consider the test function $f_b(z)$ as in Lemma 6. We get

$$\begin{aligned}
\|I_g\| &\gtrsim \|I_g f_b\|_{\mathcal{Q}_p^\beta(\mathbb{D})} \\
&= \sup_{a \in \mathbb{D}} \left(\int_{\mathbb{D}} (1 - |b|^2)^{4-4\beta} |\sigma'_b(z)|^2 |g(z)|^2 (1 - |z|^2)^{4\beta-4} (1 - |\sigma_a(z)|^2)^{p+2-2\beta} dA(z) \right)^{1/2} \\
&\geq \left(\int_{\mathbb{D}} (1 - |b|^2)^{4-4\beta} |g(\sigma_b(z))|^2 (1 - |\sigma_b(z)|^2)^{4\beta-4} (1 - |z|^2)^{p+2-2\beta} dA(z) \right)^{1/2} \\
&= \left(\int_{\mathbb{D}} |g(\sigma_b(z))|^2 |1 - \bar{b}z|^{2(4-4\beta)} (1 - |z|^2)^{p+2\beta-2} dA(z) \right)^{1/2} \\
&\gtrsim |g(b)|,
\end{aligned}$$

where we use Lemma 4.12 of [30] in the last inequality. Since b is arbitrary in \mathbb{D} , we have

$$\|I_g\| \gtrsim \sup_{z \in \mathbb{D}} |g(z)|.$$

This completes the proof. \square

Theorem 5. For $0 < p \leq 1$ and $1/2 < \beta < 1$. Let $g \in H(\mathbb{D})$. Then T_g is bounded on $\mathcal{Q}_p^\beta(\mathbb{D})$ if and only if $g \in \mathcal{Q}_p^\beta(\mathbb{D})$. Moreover, $\|T_g\| \approx \|g\|_{\mathcal{Q}_p^\beta(\mathbb{D})}$.

Proof. Take $f(z) \equiv 1$ on \mathbb{D} . It is easy to verify that $\|f\|_{\mathcal{Q}_p^\beta(\mathbb{D})} = 1$. So

$$\|T_g\| \geq \|T_g f\|_{\mathcal{Q}_p^\beta(\mathbb{D})} = \|g\|_{\mathcal{Q}_p^\beta(\mathbb{D})}.$$

Conversely, from the proof of Theorem 2.10 in [29], we know

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{2\beta-1} |f'(z)| \lesssim \|f\|_{\mathcal{Q}_p^\beta(\mathbb{D})}, \quad f \in \mathcal{Q}_p^\beta(\mathbb{D}).$$

Hence

$$\begin{aligned}
|f(z) - f(0)| &= \left| \int_0^1 f'(tz) z dt \right| \\
&\leq \|f\|_{\mathcal{Q}_p^\beta(\mathbb{D})} \int_0^1 \frac{1}{(1-t)^{2\beta-1}} dt \\
&\lesssim \|f\|_{\mathcal{Q}_p^\beta(\mathbb{D})}.
\end{aligned}$$

Then

$$|f(z)| \lesssim \|f\|_{Q_p^\beta(\mathbb{D})}.$$

We get

$$\begin{aligned} \|T_g f\|_{Q_p^\beta(\mathbb{D})} &= \sup_{a \in \mathbb{D}} \left(\int_{\mathbb{D}} |f(z)|^2 |g'(z)|^2 (1 - |z|^2)^{4\beta-4} (1 - |\sigma_a(z)|^2)^{p+2-2\beta} dA(z) \right)^{1/2} \\ &\lesssim \|f\|_{Q_p^\beta(\mathbb{D})} \|g\|_{Q_p^\beta(\mathbb{D})}. \end{aligned}$$

Thus $\|T_g\| \lesssim \|g\|_{Q_p^\beta(\mathbb{D})}$. The proof is completed. \square

From Theorems 4 and 5, we get the main result of this section.

Theorem 6. *Let $0 < p \leq 1$ and $1/2 < \beta < 1$. Then $\mathcal{M}(Q_p^\beta(\mathbb{D})) = Q_p^\beta(\mathbb{D})$.*

Proof. If $g \in Q_p^\beta(\mathbb{D})$, by the proof of Theorem 5 that g is bounded. It follows from Theorems 4 and 5 that M_g is bounded on $Q_p^\beta(\mathbb{D})$. So $g \in \mathcal{M}(Q_p^\beta(\mathbb{D}))$ and then $Q_p^\beta(\mathbb{D}) \subset \mathcal{M}(Q_p^\beta(\mathbb{D}))$.

If $g \in \mathcal{M}(Q_p^\beta(\mathbb{D}))$, take $f = 1 \in Q_p^\beta(\mathbb{D})$, then $g = gf \in Q_p^\beta(\mathbb{D})$ and $\mathcal{M}(Q_p^\beta(\mathbb{D})) \subset Q_p^\beta(\mathbb{D})$. \square

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